

DUALITY OF ANTIDIAGONALS AND PIPE DREAMS

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The cohomology ring $H^*(Fl_n)$ of the manifold of complete flags in a complex vector space \mathbb{C}^n has a basis consisting of the Schubert classes $[X_w]$, the cohomology classes of the Schubert varieties X_w indexed by permutations $w \in S_n$. The ring $H^*(Fl_n)$ is naturally a quotient of a polynomial ring in n variables; nonetheless, there are natural n -variate polynomials, the Schubert polynomials, representing the Schubert classes [LS82a]. The most widely used formulas [BJS93, FS94] for the Schubert polynomial \mathfrak{S}_w are stated in terms of combinatorial objects called *reduced pipe dreams*, which can be thought of as subsets of an $n \times n$ grid associated to w .

Reduced pipe dreams are special cases of curve diagrams invented by Fomin and Kirillov [FK96]. They were developed in a combinatorial setting by Bergeron and Billey [BB93], who called them *rc-graphs*, and ascribed geometric origins in [Kog00, KM05]. One of the main results in the latter is that the set \mathcal{RP}_w of reduced pipe dreams is in a precise sense dual to a family \mathcal{A}_w of simpler subsets of the $n \times n$ grid called *antidiagonals* (antichains in the product of two size n chains): every antidiagonal in \mathcal{A}_w shares at least one element with every reduced pipe dream, and each antidiagonal and reduced pipe dream is minimal with this property [KM05, Theorem B]. The antidiagonals were identified there with the generators of a monomial ideal whose zero set corresponds to a certain flat degeneration of the Schubert variety X_w . Geometrically, the duality meant that the components in the special fiber are in bijection with the reduced pipe dreams in \mathcal{RP}_w , which yield directly the monomial terms in \mathfrak{S}_w . It was pointed out in [KM05, Remark 1.5.5] that the proof of this duality was roundabout, relying on the recursive characterization of \mathcal{RP}_w by “chute” and “ladder” moves [BB93], along with intricate algebraic structures on the corresponding monomial ideals; our purpose here is to give a direct combinatorial explanation.

Fix a permutation $w \in S_n$, and identify it with its **permutation matrix**, which has an entry 1 in row i and column j whenever $w(i) = j$, and zeros elsewhere. We write $w_{p \times q}$ for the upper left $p \times q$ rectangular submatrix of w and

$$r_{pq} = r_{pq}(w) = \#\{(i, j) \leq (p, q) \mid w(i) = j\}$$

for the rank of the matrix $w_{p \times q}$. Let

$$l(w) = \#\{(i, j) \mid w(i) > j \text{ and } w^{-1}(j) > i\} = \#\{i < i' \mid w(i) > w(i')\}$$

be the number of inversions of w , which is called the **length** of w .

Definition 1. A $k \times \ell$ **pipe dream** is a tiling of the $k \times \ell$ rectangle by **crosses** $\begin{smallmatrix} + \\ + \end{smallmatrix}$ and **elbows** $\begin{smallmatrix} \nearrow \\ \searrow \end{smallmatrix}$. A pipe dream is **reduced** if each pair of pipes crosses at most once.

For examples as well as further background and references, see [MS05, Chapter 16]. Pipe dreams should be interpreted as “wiring diagrams” consisting of pipes entering from the west and south edges of a rectangle and exiting through the north and east edges, with the tiles $\begin{smallmatrix} \text{---} \\ | \\ \text{---} \end{smallmatrix}$ and $\begin{smallmatrix} \diagup \\ | \\ \diagdown \end{smallmatrix}$ indicating intersections and bends of the pipes.

The set \mathcal{RP}_w of reduced pipe dreams for a permutation w consists of those $n \times n$ pipe dreams with $l(w)$ crosses such that the pipes entering row i from the west exit from column $w(i)$. In such a pipe dream D , all of the tiles below the main southwest-to-northeast (anti)diagonal are necessarily elbow tiles. We identify D with its set of crossing tiles, so that $D \subseteq [n] \times [n]$ is a subset of the $n \times n$ grid.

Definition 2. An **antidiagonal** is a subset $A \subseteq [n] \times [n]$ such that no element is (weakly) southeast of another: $(i, j) \in A$ and $(i, j) \leq (p, q) \Rightarrow (p, q) \notin A$. Let \mathcal{A}_w be the set of minimal elements (under inclusion) in the union over all $1 \leq p, q \leq n$ of the set of antidiagonals in $[p] \times [q]$ of size $1 + r_{pq}(w)$.

For example, when $w = 2143 \in S_4$,

$$\begin{aligned} \mathcal{A}_{2143} &= \left\{ \{(1, 1)\}, \{(1, 3), (2, 2), (3, 1)\} \right\} \\ \text{and } \mathcal{RP}_{2143} &= \left\{ \{(1, 1), (1, 3)\}, \{(1, 1), (2, 2)\}, \{(1, 1), (3, 1)\} \right\}. \end{aligned}$$

As another example, when $w = 1432 \in S_4$,

$$\mathcal{A}_{1432} = \left\{ \{(1, 2), (2, 1)\}, \{(1, 2), (3, 1)\}, \{(1, 3), (2, 1)\}, \{(1, 3), (2, 2)\}, \{(2, 2), (3, 1)\} \right\}$$

and

$$\begin{aligned} \mathcal{RP}_{1432} &= \left\{ \{(1, 2), (1, 3), (2, 2)\}, \{(1, 2), (2, 1), (3, 1)\}, \right. \\ &\quad \left. \{(2, 1), (2, 2), (3, 1)\}, \{(1, 2), (2, 1), (2, 2)\} \right\}. \end{aligned}$$

Given any collection \mathcal{C} of subsets of $[n] \times [n]$, a **transversal** to \mathcal{C} is a subset of $[n] \times [n]$ that meets every element of \mathcal{C} at least once. The **transversal dual** of \mathcal{C} is the set \mathcal{C}^\vee of all minimal transversals to \mathcal{C} . (Our definition of transversal differs from that in matroid theory, where a transversal meets every subset only once. Here, our transversals do not give rise to matroids: the transversal duals need not have equal cardinality, so they cannot be the bases of a matroid.) When no element of \mathcal{C} contains another, it is elementary that taking the transversal dual of \mathcal{C}^\vee yields \mathcal{C} .

Our goal is a direct proof of the following, which is part of [KM05, Theorem B]; see also [MS05, Chapter 16] for an exposition, where it is isolated as Theorem 16.18.

Theorem 3. *For any permutation w , the transversal dual of the set \mathcal{RP}_w of reduced pipe dreams for w is the set \mathcal{A}_w of antidiagonals for w ; equivalently, $\mathcal{RP}_w = \mathcal{A}_w^\vee$.*

In other words, every antidiagonal shares at least one element with every reduced pipe dream, and it is minimal with this property.

Proof. We will show two facts.

Claim 1. $D \in \mathcal{RP}_w \Rightarrow D \supseteq E$ for some $E \in \mathcal{A}_w^\vee$.

Claim 2. $E \in \mathcal{A}_w^\vee \Rightarrow E \in \mathcal{RP}_v$ for some permutation $v \geq w$ in Bruhat order.

Assuming these, the result is proved as follows. First we show that $\mathcal{A}_w^\vee \subseteq \mathcal{RP}_w$. To this end, suppose $E \in \mathcal{A}_w^\vee$. Then $E \in \mathcal{RP}_v$ for some $v \geq w$ by Claim 2, so $E \supseteq D$ for some $D \in \mathcal{RP}_w$ by elementary properties of Bruhat order (use [MS05, Lemma 16.36], for example: reduced pipe dreams for v are certain reduced words for v , and each of these contains a reduced subword for w). Claim 1 implies that $D \supseteq E'$ for some $E' \in \mathcal{A}_w^\vee$. We get $E = E'$ by minimality of E , so $E = D$ and $v = w$.

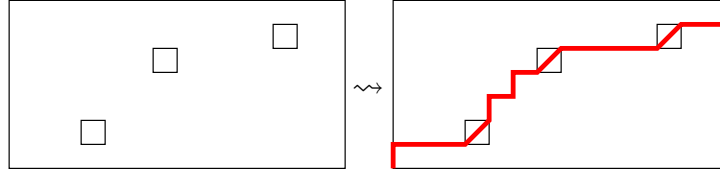
To show that $\mathcal{RP}_w \subseteq \mathcal{A}_w^\vee$, assume that $D \in \mathcal{RP}_w$. Claim 1 implies that $D \supseteq E$ for some $E \in \mathcal{A}_w^\vee$. But $E \in \mathcal{RP}_w$ by the previous paragraph, so $D = E$ because all reduced pipe dreams for w have the same number of crossing tiles.

The remainder of this paper proves Claims 1 and 2. \square

The key to proving Claims 1 and 2 is the combinatorial geometry of pipe dreams. For this purpose, we identify $[n] \times [n]$ with an $n \times n$ square tiled by closed unit subsquares, called **boxes**. This allows us to view pipes, crossing tiles, elbow tiles, and pieces of these as curves in the plane. We shall additionally need the following.

Definition 4. A **northeast grid path** is a connected arc whose intersection with each box is one of its four edges or else the rising diagonal \nearrow of the box.

Example 5. Fix an antidiagonal A in the $k \times \ell$ rectangle $[k] \times [\ell]$. There exists a northeast grid path G , starting at the southwest corner of $[k] \times [\ell]$ and ending at the northeast corner, whose sole \nearrow diagonals pass through the boxes in A . There might be more than one; a typical path G with $k = 7$, $\ell = 15$, and $|A| = 3$ looks as follows:



Example 6. Let P be a pipe in a pipe dream, or a connected part of a pipe. Define $\text{up}(P)$ to be the northeast grid path consisting of the north edge of each box traversed horizontally by P , the west edge of each box traversed vertically by P , and the rising diagonal in each box through which P enters from the south and exits to the east. Dually, define $\text{dn}(P)$ to consist of the south edge of each box traversed horizontally by P , the east edge of each box traversed vertically by P , and the rising diagonal in each box through which P enters from the west and exits to the north.



Whenever a northeast grid path is viewed as superimposed on a pipe dream, we always assume (either by construction or by fiat) that no pipe crosses it vertically through a diagonal \nearrow segment. This is especially important in the next two lemmas.

The arguments toward Claims 1 and 2 are based on two elementary principles for a region R bounded by northeast grid paths. Such a region has a lower (“southeast”) border $SE = SE(R)$ and an upper (“northwest”) border $NW = NW(R)$.

Lemma 7 (Incompressible flow). *Fix a pipe dream. If k pipes enter R vertically through SE and none cross SE again, then NW has at least k horizontal segments.*

Proof. Every pipe crossing SE vertically exits R vertically through NW . \square

Thus the “flow” consisting of the pipes entering from the south is “incompressible”.

Lemma 8 (Wave propagation). *If none of the pipes entering R vertically through SE cross SE again, then $\#\{ \diagup \text{ segments in } SE \} \geq \#\{ \diagup \text{ segments in } NW \}$.*

Proof. The sum of the numbers of horizontal and diagonal segments on NW equals the corresponding sum for SE since these arcs enclose a region. Now use Lemma 7. \square

The “waves” here are formed by the northwest halves of elbow tiles, each viewed as being above a corresponding rising \diagup diagonal; see also the proof of Lemma 11. In the proof of Proposition 12, the “flipped” version is applied: if none of the pipes entering the region R vertically (downward) through NW cross NW again, then $\#\{ \diagdown \text{ segments in } NW \} \geq \#\{ \diagdown \text{ segments in } SE \}$.

Proposition 9. *If $D \in \mathcal{RP}_w$ has no \perp on an antidiagonal $A \subseteq [p] \times [q]$ then $|A| \leq r_{pq}$.*

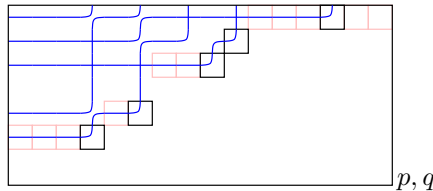
Proof. The q pipes in D that exit to the north from columns $1, \dots, q$ are of two types: r_{pq} of them enter $[p] \times [q]$ horizontally into rows $1, \dots, p$, and the other $q - r_{pq}$ of them enter into $[p] \times [q]$ vertically from the south. Now simply apply the principle of incompressible flow to the region bounded by a northeast grid path as in Example 5 and the path consisting of the south and east edges of $[p] \times [q]$. \square

Corollary 10. *Every pipe dream $D \in \mathcal{RP}_w$ is transversal to \mathcal{A}_w , so Claim 1 holds.*

Proof. If an antidiagonal $A \subseteq [p] \times [q]$ lies in \mathcal{A}_w , then by definition A has size at least $1 + r_{pq}(w)$. Now use Proposition 9. \square

Lemma 11. *If $D \in \mathcal{RP}_v$ for some permutation v , then for every $p, q \in \{1, \dots, n\}$, there is an antidiagonal of size $r_{pq}(v)$ in $[p] \times [q]$ on which D has only elbows.*

Proof. Let I_{pq} be the set of all r_{pq} of the pipes in D that enter weakly above row p and exit weakly to the left of column q . For each $k \leq q$, let b_k be the southernmost box (if it exists) in column k that intersects any $P \in I_{pq}$; otherwise, let b_k be the northernmost box in column k . Of the q pipes exiting to the north from columns $1, \dots, q$, precisely $q - r_{pq}$ of them cross some b_k vertically from the south. The remaining r_{pq} of the boxes b_k must be elbow tiles, and these form the desired antidiagonal. \square



The pipes in I_{pq} and the boxes b_1, \dots, b_q in the proof of Lemma 11

Proposition 12. *Every transversal $E \in \mathcal{A}_w^\vee$, thought of as a pipe dream, is reduced.*

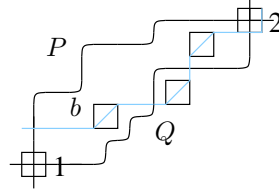


Illustration of the proof of Proposition 12

Proof. Fix a (not necessarily minimal) transversal E of \mathcal{A}_w containing two pipes P and Q that cross twice, say at \boxplus_1 and \boxplus_2 , with \boxplus_2 northeast of \boxplus_1 . Assume that the pipes P and Q as well as the crosses \boxplus_1 and \boxplus_2 are chosen so that the taxicab distance (i.e., the sum of the numbers of rows and columns) between them is minimal. Then one of the pipes, say P , is northwest of the other on the boundary of this area. The minimality condition implies that no pipe in E crosses P or Q twice, so the principle of wave propagation holds for any region R such that $SE(R)$ is part of $\text{up}(Q)$, and the flipped version holds if $NW(R)$ is part of $\text{dn}(P)$.

Our goal is to show that if \boxplus_2 is replaced by an elbow tile in E , then E will still have a crossing tile on every antidiagonal $A \in \mathcal{A}_w$, whence the transversal E is not minimal. The method: for any $A \in \mathcal{A}_w$ containing \boxplus_2 , we produce a new antidiagonal $A' \in \mathcal{A}_w$ such that $\boxplus_2 \notin A'$, and furthermore every box in A' is either an elbow tile in E or a crossing tile of A . Since A' contains a crossing tile of E other than \boxplus_2 (by construction and transversality of E), we conclude that A does, as well.

Assume that some box of A lies on \boxplus_2 . For notation, let \square_P be the box containing the only elbow tile of P in the same row as \boxplus_2 , and \square_Q the box containing the only elbow tile of Q in the same column as \boxplus_2 . Construct A' from A using one of the following rules, depending on how A is situated with respect to P and Q . (Some cases are covered more than once; for example, if the next box of A strictly southwest of \boxplus_2 lies between P and Q but south of the row containing \square_Q .)

- If the southwest box in A is on \boxplus_2 , or if A continues southwest with its next box in a column strictly west of \square_P , then move A 's box on \boxplus_2 west to \square_P .
- If A continues southwest of \boxplus_2 with its next box in a row strictly south of \square_Q , then move A 's box on \boxplus_2 south to lie on \square_Q .

For the remaining cases, we can assume that A has a box strictly southwest of \boxplus_2 but between P and Q (lying on one of P or Q is allowed). Let b be the southwest-most such box of A , and let \bar{A} consist of the boxes of A between \boxplus_2 and b .

- Assume that A continues to the west of P southwest of b . Let G be a northeast grid path passing through all the boxes in \bar{A} as in Example 5, starting with the bottom edge of the box on P that is in the same row as b , and ending with the east edge of \boxplus_2 . Applying the flipped version of wave propagation to the region enclosed by G and $\text{dn}(P)$, we conclude that we can define A' by replacing $\bar{A} \cup \{\boxplus_2\}$ with an equinumerous set of elbow tiles on P .
- If A continues to the south of Q after b , let G be a northeast grid path passing through all the boxes in \bar{A} as in Example 5, starting with the west

edge of the box on Q in the same column as b , and ending with the east edge of \boxplus_2 . Applying wave propagation to the region enclosed by G and $\text{up}(Q)$, we conclude that we can define A' by replacing $\bar{A} \cup \{\boxplus_2\}$ with an equinumerous set of elbow tiles on Q . \square

Corollary 13. *Claim 2 holds: $E \in \mathcal{A}_w^\vee \Rightarrow E \in \mathcal{RP}_v$ for some $v \geq w$ in Bruhat order.*

Proof. Bruhat order is characterized by $v \geq w \Leftrightarrow r_{pq}(v) \leq r_{pq}(w)$ for all p, q . As $E \in \mathcal{A}_w^\vee \Rightarrow E \in \mathcal{RP}_v$ for some v by Proposition 12, we get $v \geq w$ by Lemma 11. \square

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